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MHD Flow of Newtonian Fluid over a Suddenly Accelerated Flat Plate

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Abstract

In this paper the laminar flow of Newtonian conducting fluid produced by a moving plate in presence of transverse magnetic field is considered. The basic equation governing the motion of such flow is expressed in non-dimensional form. Analytic solution of the governing equation is obtained by Laplace transformation. Numerical solution of the dimensionless equation is also obtained with the help of Crank-Nicholson implicit scheme.

Key word: Viscous fluid, MHD flow, Laplace transformation, Crank-Nicholson implicit scheme.

1. Introduction

If a magnetic field is placed before a moving conducting fluid then the motion of the fluid is changed by the influence of the magnetic field. The magnetic field is also perturbed by the motion of the fluid: one affects the other and vice versa. The motion of the conducting fluid across the magnetic field generates electric-currents which changes the magnetic field and the action of magnetic field on these currents gives rise to mechanical forces which modify the flow of the fluid. The electromagnetic field is governed by Maxwell's electromagnetic equations and the motion of the fluid is governed by the field equations of the fluid mechanics. In recent years, the study of MHD phenomena in liquid conductors has received considerable impetus on account of its theoretical experimental and practical applications. Schlichting[6] studied the problem of an incompressible viscous fluid flow problem produce by the oscillation of a plane solid wall. This problem is also known as stokes second problem. Von Keregek and Davis[8] performed the linear stability theory of oscillating Stokes layers. Pauton[3] obtained the transient solution for the flow due to the oscillation plane. Erdogan[1] derived the analytic solutions for the flow produced by the small oscillating wall for small and large time by Laplace transformation method. Recently Poria, Mamaloukas, layek and Magumdar [4] derived the solution of laminar flow of viscous conduction fluid produced by the oscillating plane wall. They solved the problem both analytically and numerically in presence of magnetic field. In this paper the aim is to investigate the effects of a transverse magnetic field on the incompressible electrically conducting fluid flow produced by a moving plate. An analytic solution for the problem is presented by using Laplace transformation. The problem has been solved numerically using well known Crank-Nicholson Implicit scheme.

Formulation of the problem

A conducting viscous incompressible fluid moving in a magnetic field is governed by the following equations:

(a) Maxwell's equations

$$\nabla \cdot \overline{B} = 0 \qquad (1)$$

$$\nabla \cdot \overline{E} = \frac{\rho}{\varepsilon} \tag{2}$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{3}$$

$$\nabla \times \overline{B} = \mu_c \overline{J} \ . \tag{4}$$

(b) Navier Stokes' equation of motion

$$\frac{d\overline{V}}{dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\overline{V} + \frac{1}{\rho}(\overline{J} \times \overline{B}). \tag{5}$$

(c) Equation of continuity

$$\nabla \cdot \overline{V} = 0 \, . \tag{6}$$

In a stationary conductor Ohm's law gives

$$\overline{J} = \sigma\overline{E} \ . \tag{7}$$

In transferring to a moving frame of reference it must be remember that conductivity depends on the local state of the conducting fluid and so must be evaluated in the moving frame of reference. The modified Ohm's law is given by replacing $\overline{E}$ by $\overline{E} + \overline{V} \times \overline{B}$ in Eq.(7), i.e.,

$$\overline{J} = \sigma(\overline{E} + \overline{V} \times \overline{B}) \ . \tag{8}$$

The mechanical force of electromagnetic origin is perpendicular to the magnetic field; it has no direct influence on the motion parallel to the field. When the motion is perpendicular to the field we can write

$$\overline{J} \times \overline{B} = \sigma(\overline{E} + \overline{V} \times \overline{B}) \times \overline{B} \ . \tag{9}$$

If we consider the electrodes by highly conducting wire along the plates such a way that the potential at the two electrodes are the same, then the electric field E will be zero. Thus from Eq.(9) we have

$$\overline{J} \times \overline{B} = \sigma(\overline{V} \times \overline{B}) \times \overline{B} \tag{10}$$

or,

$$\overline{J} \times \overline{B} = -\sigma[(\overline{B} \cdot \overline{B})\overline{V} - (\overline{B} \cdot \overline{V})\overline{B}] \ . \tag{11}$$

Assume $\overline{V}$ is perpendicular to $\overline{B}$. From Eq.(11) we have

$$\overline{J} \times \overline{B} = -\sigma B^2 \overline{V} \ . \tag{12}$$

When the magnetic field is uniform and is equal to B_0, then Eq.(12) reduces to

$$\bar{J} \times \bar{B} = -\sigma B_o^2 \bar{V}. \tag{13}$$

Using Eq.(5) and Eq.(13) we get the equation of motion of the conducting fluid in presence of transverse magnetic field as

$$\frac{d\bar{V}}{dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\bar{V} - \frac{1}{\rho}\sigma B_o^2 \bar{V} \tag{14}$$

where $\bar{V}$ is the fluid velocity, p the fluid pressure, ν the kinetic coefficient of viscosity, B_0 the uniform magnetic field, σ the electrical conductivity.

Let us consider a flat plate extended to large distances in x' and z' directions. Again we consider an incompressible viscous fluid over a half plane solid wall $y' = 0$. Suppose the fluid is at rest at time $t' < 0$. At $t' = 0$ the plane solid wall $y' = 0$ is suddenly set in motion in x' direction at constant velocity U_o. As a result a two dimensional parallel flow will be produced near the plate. Since the fluid flows along x' direction and there is no velocity component along the direction perpendicular to the direction of flow, so the equation of conservation of mass reduces to

$$\frac{\partial u'}{\partial x'} = 0. \tag{15}$$

As the flow is only kept in motion by the movement of the plate, one may set the pressure gradient $\frac{\partial p'}{\partial x'} = 0$. For unsteady case Eq.(14) reduces to

$$\frac{\partial u'}{\partial t'} = \nu\frac{\partial^2 u'}{\partial y'^2} - \frac{\sigma B_o^2}{\rho}u'. \tag{16}$$

Eq. (15) indicates that u' is a function of y' and t'.

Boundary conditions:

$$\begin{aligned} &u' = 0 \text{ when } t' \le 0 \text{ for all } y' \\ &u' = U_o \text{ at } y' = 0 \text{ when } t' \ge 0 \\ &u' = 0 \text{ at } y' = \infty \text{ when } t' \ge 0. \end{aligned} \tag{17}$$

We introduce the following non-dimensional quantities

$$y=\frac{y'}{L},\ t=\frac{t'}{T},\quad u=\frac{u'}{\cup_o}$$

where L and T represent the characteristic length and characteristic time respectively .
Setting these non-dimensional quantities in Eq.(16) ,we get

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial y^2}-M^2 u \tag{18}$$

where

$$\rho\upsilon = \mu$$

$$M^2=B_0\, L\sqrt{\frac{\sigma}{\mu}}\,.$$

Here the number M is a non-dimensional number and is called Hartmann number.

In this case the boundary conditions may be written as

$$\begin{aligned} &t\le 0: u(y,\,0)=0 \qquad \textit{for all } y \\ &t\ge 0: u(0,\,t)=1\,, \qquad u(\infty,t)=0. \end{aligned} \tag{19}$$

A: Analytic Solution
We introduce the Laplace transformation and inverse Laplace transformation as

$$L\{u\,(y,t)\}=\bigcup(y,s) \tag{20}$$

$$L^{-1}\{\bigcup(y,\,s)\}=u\,(y,t)\,. \tag{21}$$

We have

$$L\left\{\frac{\partial U}{\partial t}\right\}=s\,L\{u\}\;-u\,(y,o)\;=s\bigcup \tag{22}$$

and

$$L\left\{\frac{\partial^2 u}{\partial y^2}\right\}=\frac{d^2}{dy^2}\left[L\{u\}\right]=\frac{d^2\bigcup}{dy^2}\,. \tag{23}$$

From Eq.(18), we have

$$\frac{d^2\bigcup}{dy^2}-(s+M^2)\bigcup=0\ . \tag{24}$$

With the help of boundary condition (19), we get

$$\cup(o,s) = L\{u(0,t)\} = \frac{1}{s}. \tag{25}$$

The Solution of Eq.(24) is

$$\cup(y,\, s) = c_1\, e^{y\sqrt{S+M^2}} + c_2\, e^{-y\sqrt{s+M^2}}. \tag{26}$$

Since u is finite for $y \to \infty$ we must have $c_1 = 0$.
Eq. (26) reduces to

$$\cup(y,\, s) = c_2\, e^{-y\sqrt{s+M^2}} \tag{27}$$

$$\therefore \qquad \cup\,(0,\, s) = c_2 \Rightarrow c_2 = \frac{1}{s}.$$

Thus the Eq. (26) reduces to

$$\cup(y,s) = \frac{1}{s} e^{-y\sqrt{s+M^2}}. \tag{28}$$

Taking inverse Laplace transformation of Eq. (28), we have

$$u(y,\, t) = -\frac{1}{\pi} \int_{-M^2}^{\infty} \frac{e^{-(u+M^2)t}}{u+M^2} \sin(y\sqrt{u}\,)\, du\,. \tag{29}$$

B: Numerical Solution

The Eq.(18) with initial and boundary conditions (19) is solved by finite difference technique. The Crank-Nicholson implicit scheme is used to solve the parabolic type of equation. In this scheme, the time derivative term is represented by forward difference formula while the space derivative term is represented by the average central difference formula. To do this the temporal first derivative can be approximated by

$$\frac{\partial u}{\partial t} \approx \frac{(u_i^{l+1} - u_i^l)}{\Delta\tau}. \tag{30}$$

The second derivative in space can be determined at the midpoint by the averaging the difference approximations at the beginning (t^l) and at the end (t^{l+1}) of the time increment as

$$\frac{\partial^2 u}{\partial y^2} \cong \frac{1}{2}\left[\frac{u_{i+1}^l - 2u_i^l + u_{i-1}^l}{(\Delta\eta)^2} + \frac{u_{i+1}^{l+1} - 2u_i^{l+1} + u_{i-1}^{l+1}}{(\Delta\eta)^2}\right]. \tag{31}$$

Substituting Eq. (30) and Eq. (31) into Eq. (18), we get

$$\frac{u_i^{l+1} - u_i^l}{\Delta\tau} = \frac{1}{2}\left[\frac{u_{i+1}^l - 2u_i^l + u_{i-1}^l}{(\Delta\eta)^2} + \frac{u_{i+1}^{l+1} - 2u_i^{l+1} + u_{i-1}^{l+1}}{(\Delta\eta)^2}\right] - \frac{M^2}{2}(u_i^{l+1} + u_i^l) \qquad (32)$$

or $$ru_{i-1}^{l+1} - (2r + s + 1)u_i^{l+1} + ru_{i-1}^{l+1} = (2r - 1 + s)u_i^l - r(u_{i-1}^l + u_{i+1}^l) \qquad (33)$$

where

$$r = \frac{\Delta\tau}{2(\Delta\eta)^2}, \quad s = \frac{\Delta\tau M^2}{2}.$$

The Eq.(33) can be written as

$$-ru_{i-1}^{l+1} + k_1 u_i^{l+1} - ru_{i-1}^{l+1} = r(u_{i-1}^l + u_{i+1}^l) + k_2 u_i^l \qquad (34)$$

where

$$k_1 = 1 + 2r + s \qquad \textit{and} \qquad k_2 = 1 - 2r - s.$$

The system of algebraic equations in tri-diagonal form is solved by Thomas algorithm for each time level. In this problem some grid points have been consider for numerical computation. u is obtained at each grid points at each time interval.

The figure below is drawn for various values of y when t=0.4.

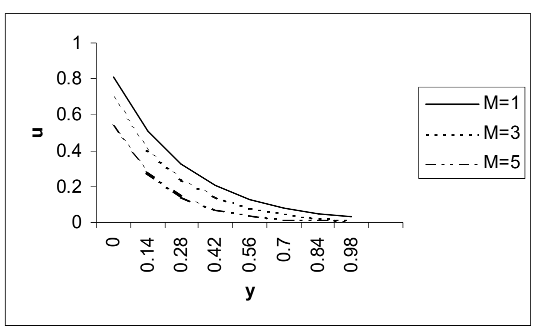

Figure: *Velocity profile for different values of Hartmann number M*

2. Result and discussion

The velocity of Newtonian conducting fluid produced by a suddenly accelerated flat plate in presence of transverse magnetic field has been derived by using analytic and numerical technique. Due to the fluid viscosity, the velocity of the fluid contact to the plate is almost same as the velocity of the plate. Since the fluid has set in motion only the moving plate, at the large distance from the plate the fluid velocity reduces to zero. In presence of transverse magnetic field the velocity of the conducting fluid is disturbed significantly. The velocity profiles are drawn for various Hartmann numbers. This figure shows that the velocity of the fluid decreases as the magnetic field increases. The velocity decreases gradually and attains almost zero velocity at a sufficient large distance from the plate.

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